Assignment 5.

This assignment is due April 25th. If you need more time, ask for an extension (just don't get overwhelmed by homeworks piling up).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

For reference: first 5 problems deal with isomorphisms, linear functionals and dual spaces (chapter 3 in the textbook); the remaining problems deal with characteristic values and other bits of theory of a single linear operator (to clarify, none of these problems require Jordan Form theorem to solve).

(1) Let $F^{n\times n}$ be the vector space of $n\times n$ matrices over a field F. Let $A \in F^{n\times n}$ be a fixed invertible matrix. Prove that $\varphi : F^{n\times n} \to F^{n\times n}$ s.t.

$$U \to A^{-1}UA$$

for any $U \in F^{n\times n}$ is an isomorphism (as a vector space) of $F^{n\times n}$ onto itself.

(2) Let V be the vector space of all polynomials p with real coefficients which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.$$

Show that (f_1, f_2, f_3) is a basis for V^* by exhibiting the basis for V of which it is the dual.

(Hint: one way is to express f_i through coefficients c_0, c_1, c_2.)

(3) Recall that trace of an $n \times n$ matrix A is the sum of its main diagonal elements: $\operatorname{tr} A = A_{11} + A_{22} + \cdots + A_{nn}$. Let $A, B \in F^{n\times n}$, where F is a field.
(a) Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
(b) Show that similar matrices have the same trace.
(c) Let $F = \mathbb{C}$. Show that there are no matrices $X, Y \in F^{n\times n}$ such that $XY - YX = I$. (Hint: compute trace of both sides.)

(4) Let W_1 and W_2 be subspaces of a finite-dimensional vector space V.
(a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.
(b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

(5) Let V be the space of all $n \times n$ matrices over a field F and let B be a fixed $n \times n$ matrix. If T is a linear operator on V defined by $T(A) = AB - BA$, and f is the trace function, what is $T^t f$?

(6) Find the characteristic polynomial, the eigenvalues, and the associated eigenvectors of this (complex) matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(7) Recall that similar matrices have the same eigenvalues. Show that the converse does not hold.

(8) Prove that if T is an invertible matrix and has eigenvalues $\lambda_1, \ldots, \lambda_n$, then T^{-1} has eigenvalues $1/\lambda_1, \ldots, 1/\lambda_n$. Is the converse true?

—— *see next page* ——

(9) (a) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k.
(b) What is wrong with this proof generalizing that? "If λ is an eigenvalue of A and μ is an eigenvalue for B, then $\lambda\mu$ is an eigenvalue for AB, for, if $Ax = \lambda x$ and $Bx = \mu x$ then $ABx = A\mu x = \mu Ax = \mu\lambda x$"?

(10) Diagonalize.

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$$

(11) For each map below, give the chain of rangespaces $R(T^k)$ and the chain of nullspaces $N(T^k)$, and the generalized rangespace and the generalized nullspace.
(a) $T : P_2 \to P_2$ (here P_2 is the space of real polynomials of degree ≤ 2), s.t. $a + bx + cx^2 \to a + cx^2$.
(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, s.t.

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} 0 \\ a \end{pmatrix}.$$

(c) $T : P_2 \to P_2$, s.t. $a + bx + cx^2 \to b + cx + ax^2$.
(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$, s.t.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} a \\ a \\ b \end{pmatrix}.$$

(12) Let $V = \mathrm{R}[x]$, the space of all polynomials with real coefficients. Let D be the differentiation operator. Find the chain of rangespaces D^kV and the chain of nullspaces $N(D^k)$. Observe that rangespaces stay the same, while nullspaces do not ever stabilize.

(13) Suppose two operators $T, S : V \to V$ are such that $TS = ST$. Prove that TV and $N(T)$ is invariant under S.